SELECTED 2009 THE ROYAL SOCIETY SUMMER SCIENCE EXHIBITION

How do shapes fill space?

How do Triangles meet?

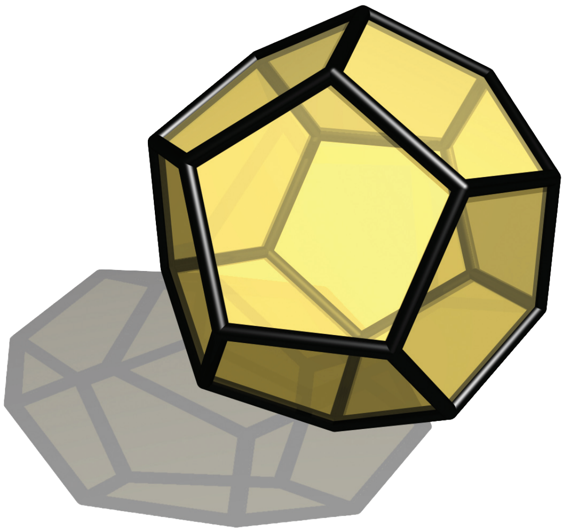
Factsheet 3

Going round a vertex...

For tilings using regular polygons we looked at what we could fit round a vertex. We wanted the total angle round each vertex to give a full circle. In the poster we explored what happened with equilateral triangles as we added and removed them. Here we look in more detail and explore other shapes.

Less than a circle (spherical)

Start with pentagons. These were somehow missed out in the Platonic tilings, being the shape where we could fit three round a point, but could make no tiling. Take three pentagons and connect them together. Now pull the two unattached edges together. This pulls the pentagons out of the plane into three dimensions. At each free vertex now connect more pentagons, so each vertex has three around it. Continue until you cannot go any further. You should now have a shape with 12 pentagons as faces. This is the dodecahedron. Unlike the flat tilings this is a finite object, and it is three dimensional. In fact it is a three dimensional version of the regular polygons, as every vertex and face are the same:



In geometry if we have less than 360° at an angle we get spherical geometry, which is why the dodecahedron closes up. Look around, you can find other example of Archimedean polyhedra where each vertex is the same (though they may have different faces).

Classification of the regular polyhedra

The ancient Greeks considered this problem and were able to prove that there were exactly five such shapes in three dimensions. On the poster and above we have now come accross four of these. Can you work out what the final one is? To prove this they used similar methods to the proof on Fact sheet 1. However, instead of considering shapes where the angle added to a full circle, they just looked for shapes where the angle was less.

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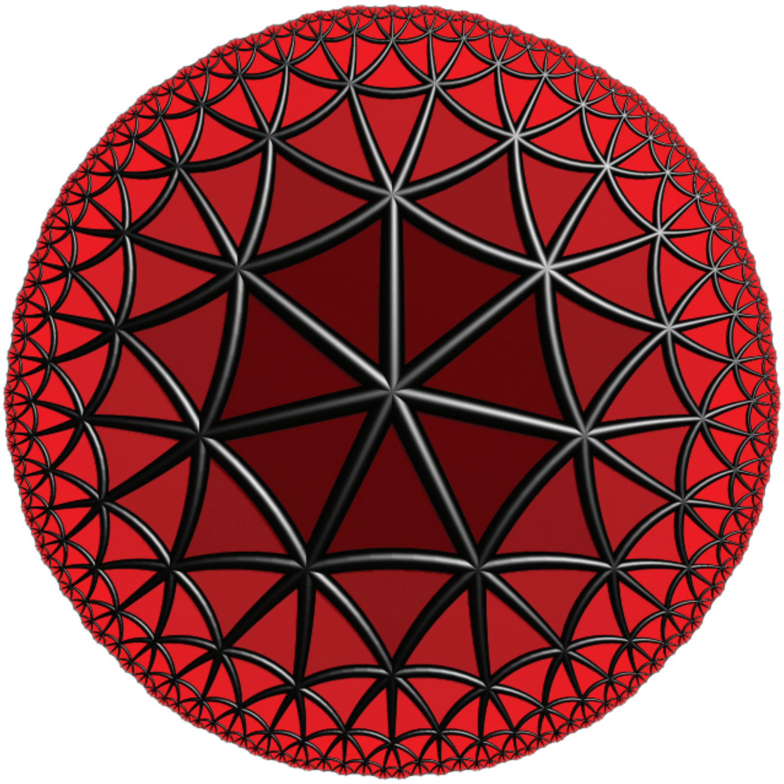
More than a circle (hyperbolic)

Now go back to triangles. Think about putting three round a vertex, then four, then five (as above), when we get to six we get to a flat tiling. Now try seven. Its not very hard, the tiles fold again. This gives seven new points. Put triagles around each one. As you continue you get a single sheet, but one that folds and flexes in strange ways. This is a related to *hyperbolic space*, a strange mathematical world where the angles round a point add up to more than 360. It has many strange features. There are triangles with infinitly long edges, for example.

Here is another view of this space, you can see the tiling with seven triangles round a point. In this case, "straight" lines become bent, rather than edges bending into three dimensions. Although it is infinite, this model fits it into a circle. To do this we have to bend and squash the shapes. In fact in the image all the triangles are the same shape, they only appear different in our visualisation.



Classification of geometries

Parallel lines are a key difference between the three geometries. We choose any point P and any line L. In Spherical geometry there is no line through P that does not cross L. In Euclidean geometry there is a unique line through P that does not cross L and in Hyperbolic geometry their are an infinite number. Euclidean geometry is named after Euclid as it is the geometry described in his *Elements*. In that book one of the five fundamental facts, or *axioms* used to describe the geometry was the parallel postulate, stating that there is a unique line through P that does not cross L. For thousands of years people tried to prove that this was unecessary and was a consequence of the other four axioms. Only in the 19th century, with the discovery of hyperbolic geometry, was it revealed that this quest was fruitless, as it obeys all of Euclid's axioms apart from the parallel postulate.

So what other geometries can occur that obey Euclid's first four axioms? In fact we have them all here. Spherical, Euclidean and Hyperbolic. Can you work out why there is no geometry that has exactly two lines through P that do not cross L? This is true for any number of dimensions not just two.

Today physicists look at the universe and try to establish which of the three dimensional geometries it has.

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