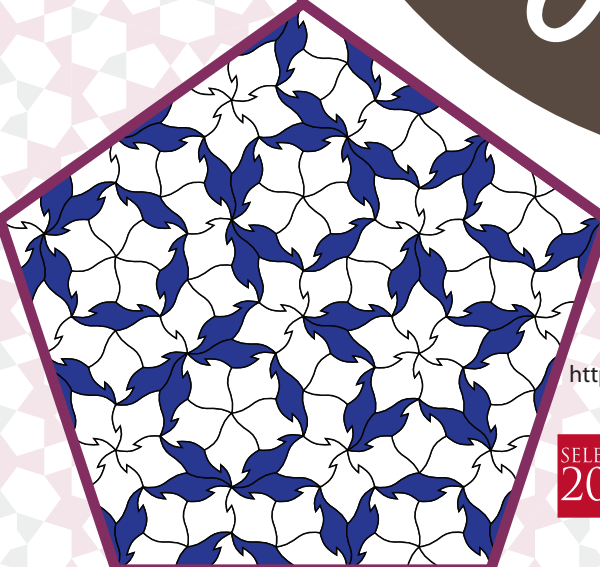
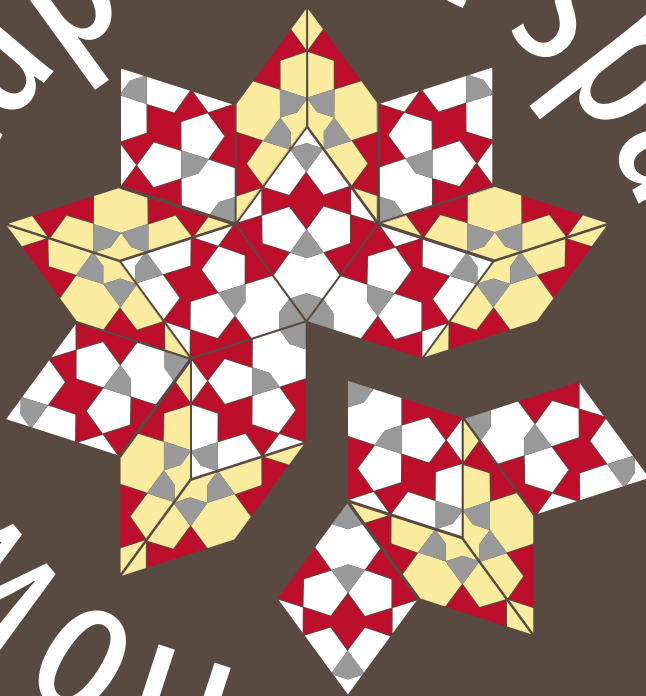


# How do shapes fill space?



30 June - 4 July 2009 at the  
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# Where do we start?

## Factsheet 1

### Proof (don't be afraid)

The wonderful thing about mathematics is that it deals with simple ideas. For anyone who has sat through a maths class (even us) this sounds like a silly statement. Let me explain. Consider something that most people would say was simple. Like hair colour. Find someone around you and say their hair colour to yourself. Do you think everyone will agree with you? Now try to find someone whose hair might cause debate, perhaps a dark blonde, many people would call brown. Now think of a country like China where there is far less variation in hair colour. Do people notice more subtle differences? Do they notice hair colour at all? You see hair colour is quite a complicated idea when we try to lift it above personal experience. Mathematics is different, we try to consider objects where every feature is defined. We cannot disagree on the meaning of the number "one" and as a result we can prove things. A correct proof is really an argument that cannot be refuted. Proof however can do more, it can take a seemingly complicated situation and make it understandable, or at least less complicated.

### Platonic Tilings

Around you you can find some regular shapes. Triangles, squares, pentagons, hexagons...go and have a play with them. Now consider the following question, how many tilings can I make using copies of just one shape? You probably know one already the tiling of squares on your bathroom wall. There are two others. Equilateral triangles and regular hexagons. That is all. How can we be sure of this? We need a proof!

Start with a shape (any regular polygon) and fit tiles round a corner. Take a square for example. Four squares fit exactly round a corner. With a pentagon, however, we have a slight gap after fitting three together. Mathematically we take the angle at a corner. This gives the following:

Shape	Corners	Angle (°)	
		Degrees	Circle fraction
Equilateral triangle	3	60	1/6
Square	4	90	1/4
Regular Pentagon	5	108	3/10
Regular Hexagon	6	120	1/3
Regular Heptagon	7	128.57	5/14
Regular Octagon	8	135	3/8
Regular Nonagon	9	140	7/18
Regular Decagon	10	144	2/5
Regular Enneagon	11	147.27	9/22
Regular Dodecagon	12	150	5/12

We can actually read off from this chart that of these shapes only the equilateral triangle, square and hexagon will fit exactly round a vertex. What about other shapes? Could things suddenly

work again for a shape with 1531 sides? The answer is no, to do this let us consider the general angle. We have to turn an equal angle at each corner, so at each corner of a shape with  $n$  sides we turn  $1/n$  (or  $360/n$  degrees). The angle inside the shape is therefore the angle of a straight line ( $1/2$  or  $180$  degrees) minus the angle we turned through. We therefore have  $1/2 - 1/n$  ( $180 - 360/n$  if you want to think in degrees). This number can only increase but can never get to  $1/2$ . As we have already passed  $1/3$  for the hexagon no other shape will work.

We are not quite finished. We have shown what cannot work, we now need to show that the three shapes do give us tilings. Can you construct the tilings with 6 equilateral triangles, 4 squares and 3 hexagons round a corner and show they will go on for ever?

**Q1** Actually this proof is a bit of a cheat. We left out a key condition, can you find other tilings with these shapes?

**Q2** What is the condition we left out?

## Generalising

Mathematicians love to take something they understand and push it further. You will see a lot of this on the next posters. We will start gently:

Look at the interior angles listed in the table. There are many ways that we can use different ones to add to 1. For example we can take two pentagons ( $3/10 \times 2 = 3/5$ ) and add a decagon ( $2/5$ ). Some of these can be used to give tilings, others can't. Some other examples are below.

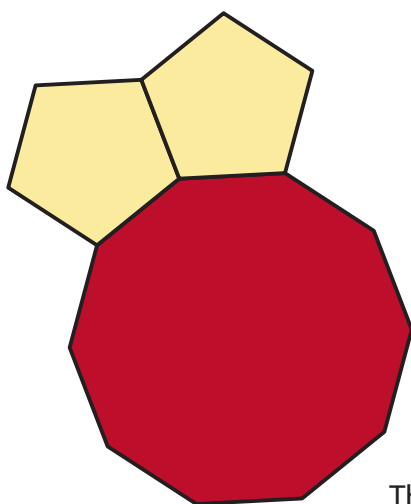
Use the tiles to experiment and answer the following questions, pictures are good here!

**Q3** Explain how a square, two triangles and dodecagon can tile the plane.

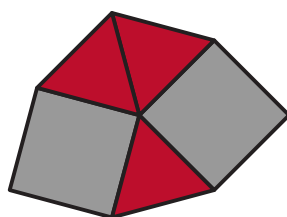
**Q4** Explain why a tiling with a decagon and two pentagons around every vertex is not possible.

**Q5** Are there any tilings with nonagons and the same tiles round every vertex.

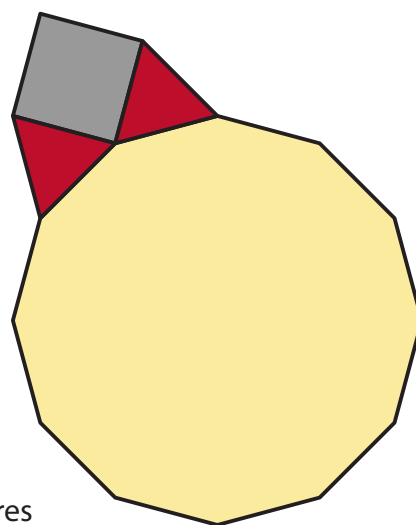
Challenge: Can you find all the tilings with the same configuration at each vertex. (Hint, start by finding all the ways to add the circle fractions add to 1, think about whether the table above is sufficient.)



Two pentagons and a decagon



Three triangles and two squares



Two triangles, a square and a dodecagon

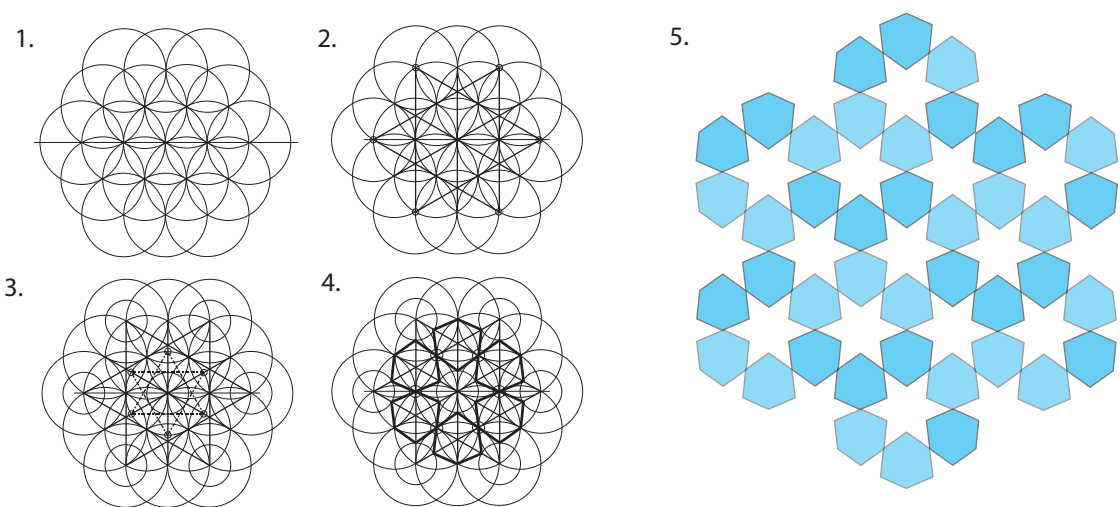


# What are the secrets of the Islamic master craftsmen?

## Factsheet 2

### Geometry in Islamic Art

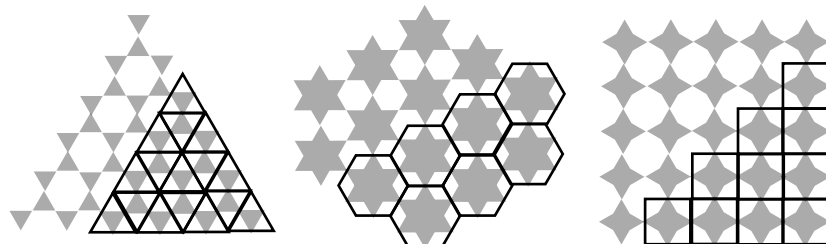
The use of geometry creates a natural harmony within Islamic art. This relates to the Islamic belief that all creation is harmoniously interrelated. In common with other traditional arts these geometric designs can be created using the simple tools of compass and straight edge. Notice how the pattern below is derived from regular divisions of the circle. A regular grid of triangles is established, on top of which the design is elaborated. Note the different stages of the pattern and how this design differs subtly from the one on the main exhibition poster.



In this example it is easy to see how one might alternatively have used a regular grid of hexagons to develop a different design. How might you construct a regular grid of squares using just compass and straight edge?

### Regular Tilings

There are just three regular tilings (sometimes referred to as the Platonic Tilings), in which the same regular polygon tiles with copies of itself to fill the plane. Many Islamic patterns are based on these three tilings, however the underlying grid of triangles, squares or hexagons is hidden in the final design. Can you see what the underlying tiling unit is in this *alicatado* (tile mosaic) pattern from the Alhambra (right)?

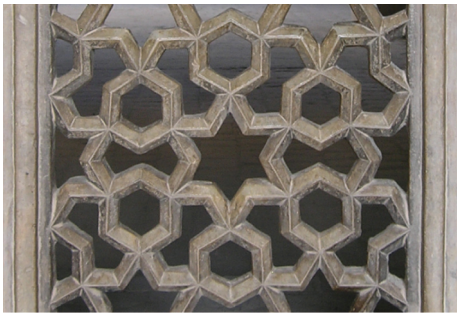
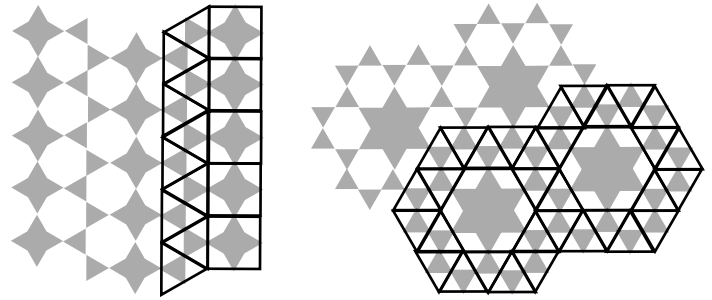


*Alicatado* (ceramic cut tiles), Alhambra, Spain.

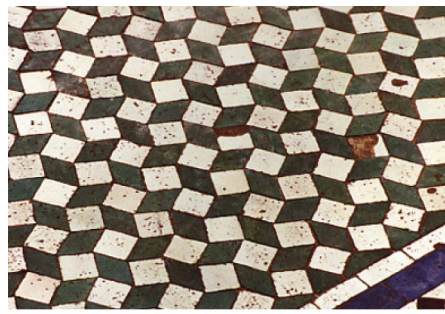
Three classical patterns shown with and without the underlying tiling.

# Semi-Regular Tilings

On Factsheet 1 you were asked to find all possible tilings in which the same regular polygons meet at each vertex. These are known as the semi-regular tilings and they are hidden beneath more complex Islamic patterns. Two examples are illustrated (right). Can you find the underlying tiling in the patterns below?



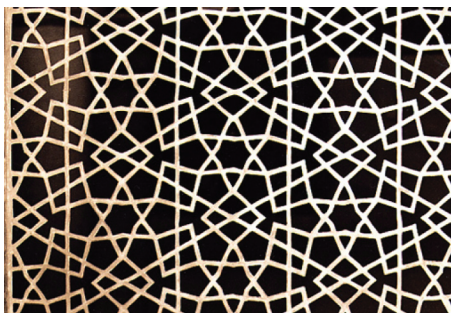
Pierced stone screen, Isfahan Iran.



Ceramic floor tiles, Morocco.



Ceramic *zillij* (cut-tile), Meknes, Morocco.



Iron grill, Ibn Tulun mosque, Cairo.



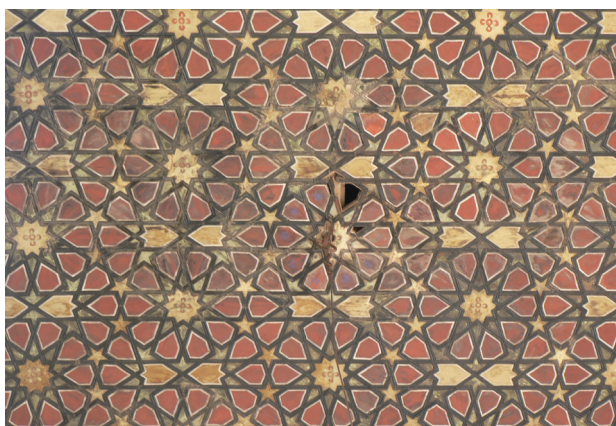
Cut-stone floor, Cairo (Photo: P Marchant).



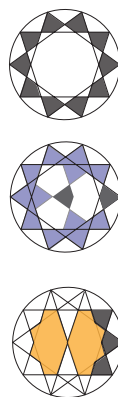
Cut-tile mosaic, Kerman, Iran.

# Non-Archimedean Tilings

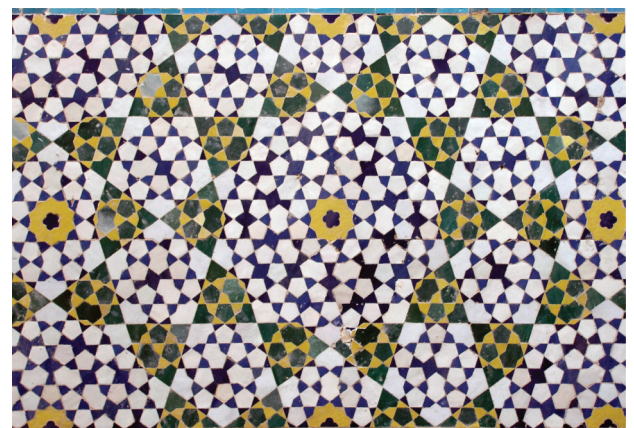
As you will have observed, not all polygons will tile the plane. The pentagon presents particular problems, but Islamic designers showed great ingenuity in overcoming these. The pattern below left is a slight variation on one that you can create with the wooden tiles. What additional tile/tiles would you need to create this pattern? The pattern below right is based on a similar underlying grid, but appears more complex as it employs different breakdowns of the decagram motif (below centre) at each vertex in the underlying grid.



Painted ceiling, Ali Qapu palace, Isfahan, Iran.



Decagram motifs.



Ceramic tile mosaic, Vakil mosque, Shiraz, Iran.

Unless otherwise indicated, all photos and drawings by Richard Henry [www.richardhenry.info](http://www.richardhenry.info)



# How do Triangles meet?

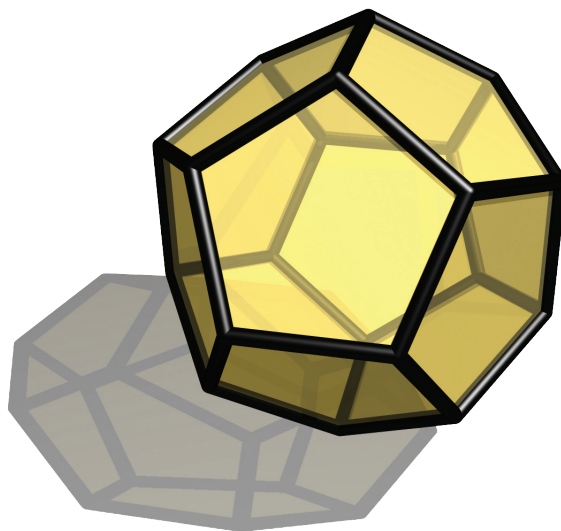
## Factsheet 3

### Going round a vertex...

For tilings using regular polygons we looked at what we could fit round a vertex. We wanted the total angle round each vertex to give a full circle. In the poster we explored what happened with equilateral triangles as we added and removed them. Here we look in more detail and explore other shapes.

### Less than a circle (spherical)

Start with pentagons. These were somehow missed out in the Platonic tilings, being the shape where we could fit three round a point, but could make no tiling. Take three pentagons and connect them together. Now pull the two unattached edges together. This pulls the pentagons out of the plane into three dimensions. At each free vertex now connect more pentagons, so each vertex has three around it. Continue until you cannot go any further. You should now have a shape with 12 pentagons as faces. This is the dodecahedron. Unlike the flat tilings this is a finite object, and it is three dimensional. In fact it is a three dimensional version of the regular polygons, as every vertex and face are the same:



In geometry if we have less than  $360^\circ$  at an angle we get spherical geometry, which is why the dodecahedron closes up. Look around, you can find other example of Archimedean polyhedra where each vertex is the same (though they may have different faces).

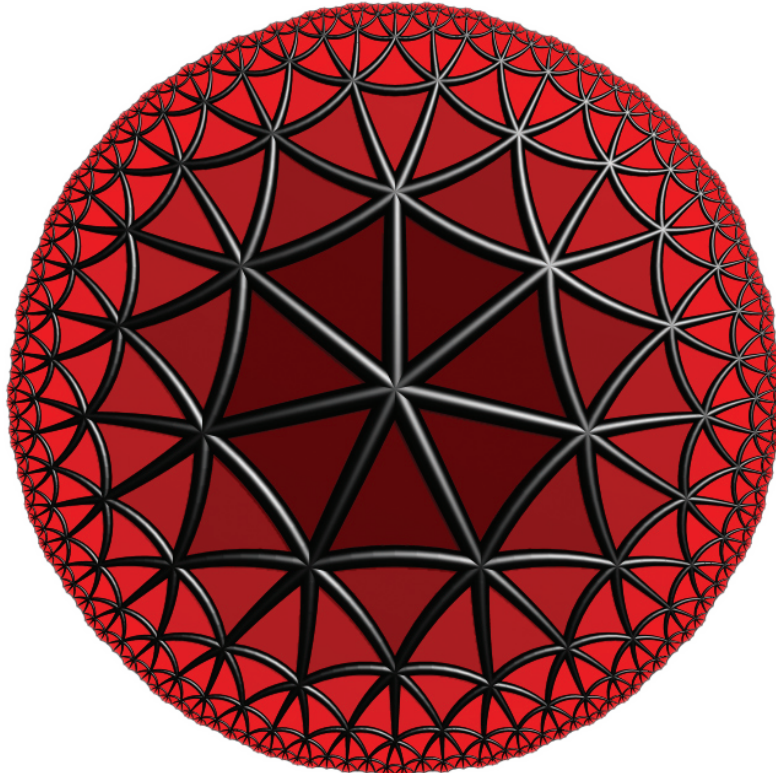
### Classification of the regular polyhedra

The ancient Greeks considered this problem and were able to prove that there were exactly five such shapes in three dimensions. On the poster and above we have now come accross four of these. Can you work out what the final one is? To prove this they used similar methods to the proof on Fact sheet 1. However, instead of considering shapes where the angle added to a full circle, they just looked for shapes where the angle was less.

## More than a circle (hyperbolic)

Now go back to triangles. Think about putting three round a vertex, then four, then five (as above), when we get to six we get to a flat tiling. Now try seven. Its not very hard, the tiles fold again. This gives seven new points. Put triagles around each one. As you continue you get a single sheet, but one that folds and flexes in strange ways. This is a related to *hyperbolic space*, a strange mathematical world where the angles round a point add up to more than 360. It has many strange features. There are triangles with infinitely long edges, for example.

Here is another view of this space, you can see the tiling with seven triangles round a point. In this case, "straight" lines become bent, rather than edges bending into three dimensions. Although it is infinite, this model fits it into a circle. To do this we have to bend and squash the shapes. In fact in the image all the triangles are the same shape, they only appear different in our visualisation.



## Classification of geometries

Parallel lines are a key difference between the three geometries. We choose any point  $P$  and any line  $L$ . In Spherical geometry there is no line through  $P$  that does not cross  $L$ . In Euclidean geometry there is a unique line through  $P$  that does not cross  $L$  and in Hyperbolic geometry there are an infinite number. Euclidean geometry is named after Euclid as it is the geometry described in his *Elements*. In that book one of the five fundamental facts, or *axioms* used to describe the geometry was the parallel postulate, stating that there is a unique line through  $P$  that does not cross  $L$ . For thousands of years people tried to prove that this was unnecessary and was a consequence of the other four axioms. Only in the 19th century, with the discovery of hyperbolic geometry, was it revealed that this quest was fruitless, as it obeys all of Euclid's axioms apart from the parallel postulate.

So what other geometries can occur that obey Euclid's first four axioms? In fact we have them all here. Spherical, Euclidean and Hyperbolic. Can you work out why there is no geometry that has exactly two lines through  $P$  that do not cross  $L$ ? This is true for any number of dimensions not just two.

Today physicists look at the universe and try to establish which of the three dimensional geometries it has.



# How do 3d shapes meet?

## Factsheet 4

### Going round an edge...

We have been looking at placing two dimensional tiles round a corner. This has given tilings on the plane and shapes in three dimensions. What happens when we want to put these together? Think about tetrahedra. Take one and stick a second to its face, now stick on a third, and so on. We are now not going round a corner, we are going round an edge. How many tetrahedra can we place round an edge? We need the angle between two faces. If you are familiar with sine and cosine you can work this out, but to cheat here are the answers for all five regular polyhedra:

Shape	Faces	Edges	Corners	Dihedral angle	
				Degrees	Circle fraction
Tetrahedron	4	6	4	70.53	0.196
Cube	6	12	8	90	0.25
Octahedron	8	12	6	109.47	0.304
Dodecahedron	12	30	20	116.56	0.324
Icosahedron	20	30	12	138.19	0.384

You see from this that five tetrahedra fit round an edge but do not meet up as 0.196 is slightly less than  $1/5 = 0.2$ . Keep this in mind, as you will remember we have a way to deal with this. For the moment we see that the only shape that will meet up is the cube, that meets itself after four go round an edge. Look closer though. the tetrahedron and the octahedron have angles that add to 180, or half a circle (using sine and cosine you can prove that this is exact). In three dimensions therefore we can make a tiling with four cubes round an edge or two octahedra and two tetrahedra.

### The fourth dimension

Remember how we got from two dimensions to three dimensions? We took something that did not quite fit when flat (tiles round a point) and folded. We can do the same thing to three dimensional polyhedra that do not quite fill round an edge! The idea of the fourth dimension has entered the popular imagination and has been written about far outside mathematics and science. Unfortunately, much of what has been written is rubbish. So what is the fourth dimension to a mathematician? It is simply a fourth number (coordinate) to describe a point in space. That number could be time, but in many cases, including this exhibit, it is not. Yet this simple space is home to marvels.

The trouble with this space is that we cannot see it visually. In fact we only see three dimensions using some clever tricks. We all combine the two 2-dimensional images from our eyes into a 3d world. The three dimensional world we live in is converted to a two dimensional image using a process called projection. Think about an object casting a shadow in sunlight. Any two points on a straight line with the sun will cast the same shadow. Using a similar method we can show three dimensional shadows of four dimensional objects. The printed images on the poster and here are projections to 2d of 3d models of 4d objects. Even though they are flat we can see the 3d structure thanks to the tricks our eyes perform.

So from the table we can work out what edge configurations are possible. We can have three four or five tetrahedra round an edge, three or four cubes (with four meeting perfectly) and three octahedra or dodecahedra. Only two icosahedra fit round an edge so these will fold flat. Can you figure out which of these edge configurations corresponds to which polytope on the poster?

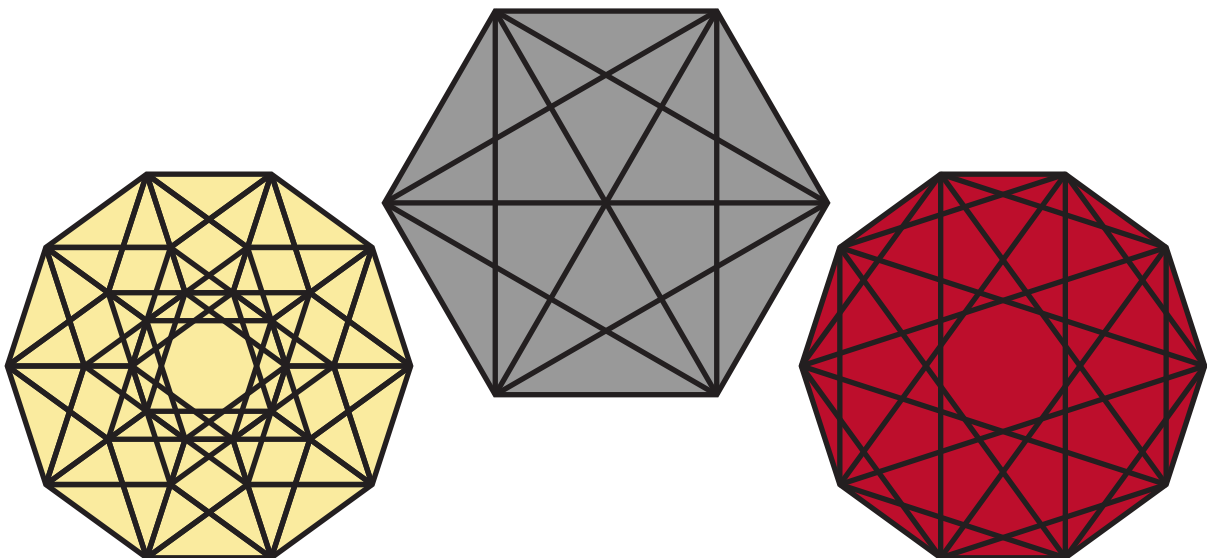
## Higher dimensions still

In two dimensions there are an infinite number of regular polygons, in three dimensions there are five regular polyhedra and now in four dimensions we find six regular shapes, called *4d-polytopes*. What about the even harder to visualise fifth dimension, or the 45th? Can we use abstract thinking to explore these higher dimensions where our intuition and visual abilities break down? In this case we can. In order to prove what polyhedra and polytopes could occur we developed a framework. We do not need to be able to play with the shapes to consider the angles, similarly to step from three to four we worked in the abstract. We can continue in the abstract. We can now look at the interior angles between three dimensional faces (*cells*) in the 4d polytopes. If you want a real challenge try to work out how! We then get the following table:

Shape	3d Cells	Faces	Edges	Corners	Dihedral angle	
					Degrees	Circle fraction
5-cell	5	10	10	5	70.56	0.196
Tesseract	8	24	32	16	90	1/4
16-cell	16	32	24	8	120	1/3
24-cell	24	96	96	24	120	1/3
120-cell	120	720	1200	600	144	2/5
600-cell	600	1200	720	120	164.52	0.457

We see from this that we can find regular tilings in 4d with Tesseracts, 16-cells and 24-cells. We can now go up to 5d. In a move that hurts your brain if you try to visualise it we now look at shapes going round a 2d face. The only shapes that can have three round a face without meeting are the 5-cell and tesseract, with 3 or 4, 5-cells and 3-tesseracts possible. This means that there are only 3 regular polytopes in 5 dimensions. In fact this is the case for every higher dimension!

To conclude here are the 2d shadows of the three regular polytopes in 5 dimensions:





# What do you see?

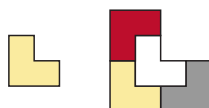
## Factsheet 5

### The Penrose Tiling

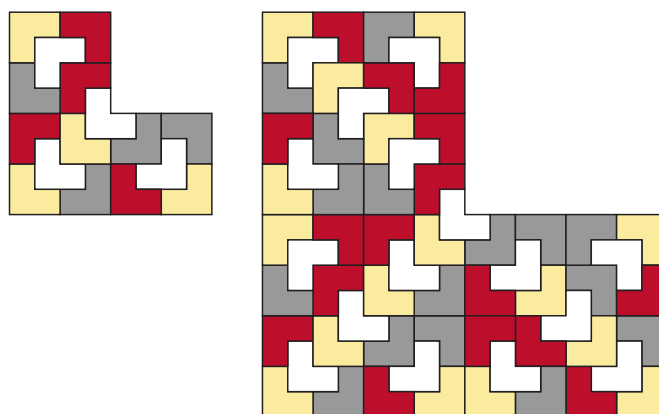
These two shapes have a remarkable property. They can tile the plane but not periodically. In other words the tiling of the plane they produce is not made of a single patch of tiles repeated. We hope here to show you the proof of this fact, remember a proof is in many ways just a simple explanation. To do that we need to do two things. Firstly we need to show that the tiles can in fact tile the plane. Secondly we need to show that no periodic tiling is possible. To show that they tile the plane we need a new construction called a substitution rule.

### Substitution Rule

Think about this L-shaped tile, we can double it in size and then cut this larger shape into four copies of the original:



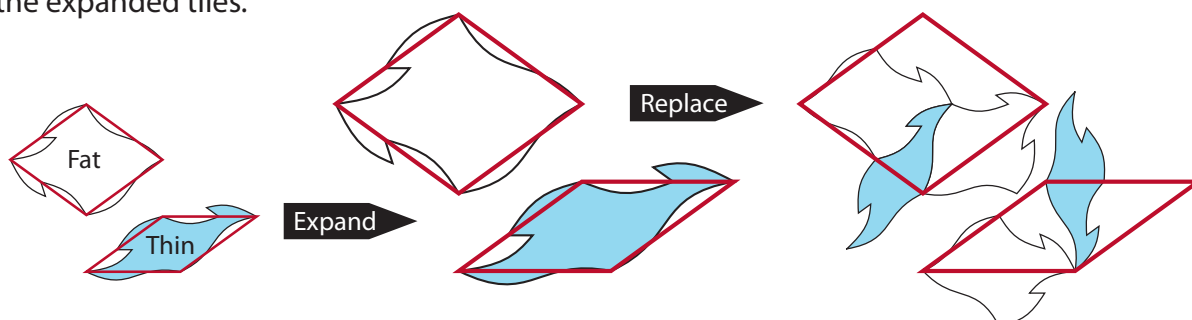
We can now repeat...



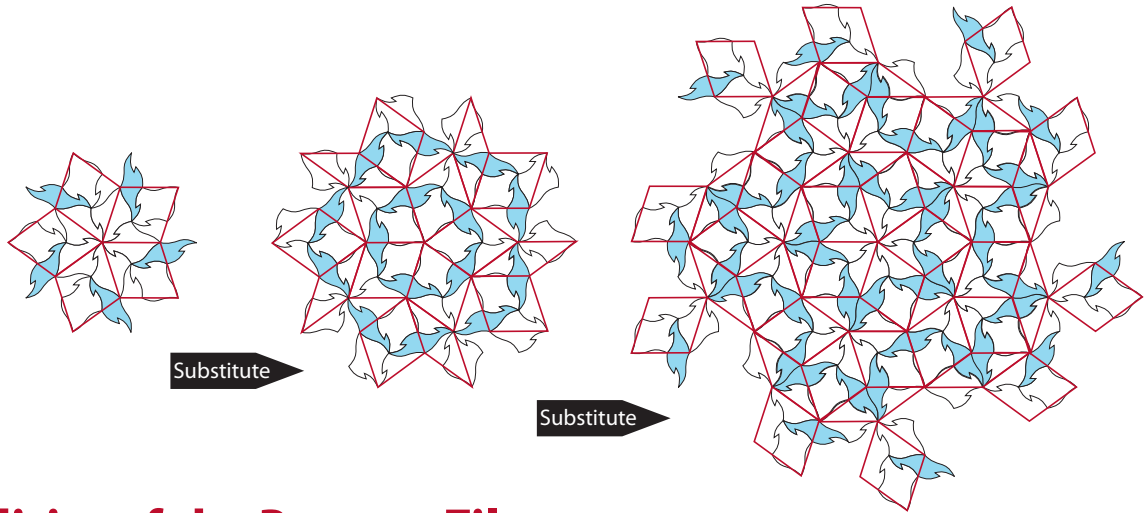
This is a simple example of a substitution rule. In general a substitution rule takes a patch of tiling, expands it and then replaces the larger tiles by patches of the original ones. So in this case we take the L-shaped tile, expand it by a factor of two and then replace it by four copies.

### Penrose Substitution

The substitution rule for the Penrose tiles is a little more complicated. In this case we expand the tiles by the golden ratio:  $(1+\sqrt{5})/2$ , and then replace the larger tiles by the following two patches. Note that these patches of tiles fit together in exactly the same way (though with different edges) as the expanded tiles.



Notice that these two patches can fit together in exactly the same way as the original two tiles. We can therefore think of them as larger versions of the original tiles. We can repeat this to get larger and larger patches of Penrose tiles:



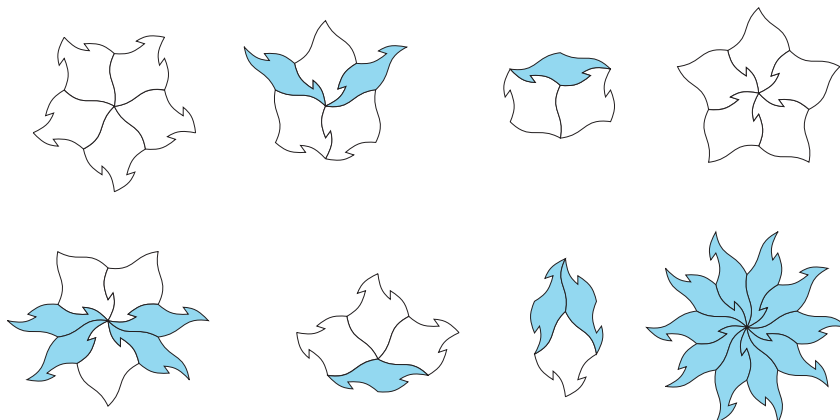
## Aperiodicity of the Penrose Tiles

We now know that the Penrose tiles can fill the plane. Can this tiling be periodic? What happens when we start with a single fat rhomb and start to apply the rule. After one application we have 2 fat rhombs and 1 thin rhomb. At each stage we replace the fat rhombs again by 2 fat and 1 thin, and the thin rhombs by 1 fat and 1 thin. So the numbers of fat and thin rhombs (thin, fat) go as follows:

(0,1), (1,2), (3,5), (8, 13), (21, 34), (55,89), (144, 233)...

This sequence might be familiar to you, these are the Fibonacci numbers, where each number is the sum of the two before. The ratio of a Fibonacci number to its predecessor gets closer and closer to the golden ratio  $(1+\sqrt{5})/2:1$  as the numbers get larger. This is therefore the ratio of fat rhombs to thin rhombs in a tiling generated by the substitution rule (as this involves applying the substitution infinitely often). This is an irrational number, one that cannot be written as a fraction. Think of a periodic tiling: it has one patch repeated, so the ratio of the tiles must be the same as the ratio in the patch. Thus the tiling given by this substitution rule cannot be periodic.

So the Penrose tiles can tile, in a non-periodic way. Is there another way? No. Lets think about how they can fit round a point, some are shown below (we have done a little of the work for you and ruled out a few that cannot continue). Can you show that these patches will either occur in the substitution tiling, or cannot be continued? Try it using the shapes provided.



# What shapes fill the plane?

## Factsheet 6



### A History of Aperiodicity

In the early 1960's the mathematician Wang was thinking about the question:

**If you are given a set of shapes of tiles, can you use them to tile the whole plane?**

Wang thought that there should be an algorithm or computer program that would be able to decide, for any set of shapes of tiles you gave it, whether or not you could use them to fill the plane. He was assuming however that if you could fill the plane with your tiles, then it would have to be by using them to form a patch that could be endlessly repeated, just as we saw on the poster with the Myers tile.

If it had been true that any tiling of the plane had to be the repetition of some basic patch, then Wang's algorithm would have gone as follows:

- Take one tile shape. Does it form a patch that can be made to repeat?  
If it does, you've got your answer. If not, test each other shape.
- If all the shapes fail, try all possibilities of 2 shapes put together.  
If one of them forms a patch that repeats then you're finished.  
If not, test all ways of putting 3 shapes together, and so on.

**Either** – you eventually get to a patch that can be made to repeat:

you can tile the plane with these shapes of tiles;

**Or** – you get to a point where you can't fit any more tiles together:

you can't tile the plane with these shapes of tiles.

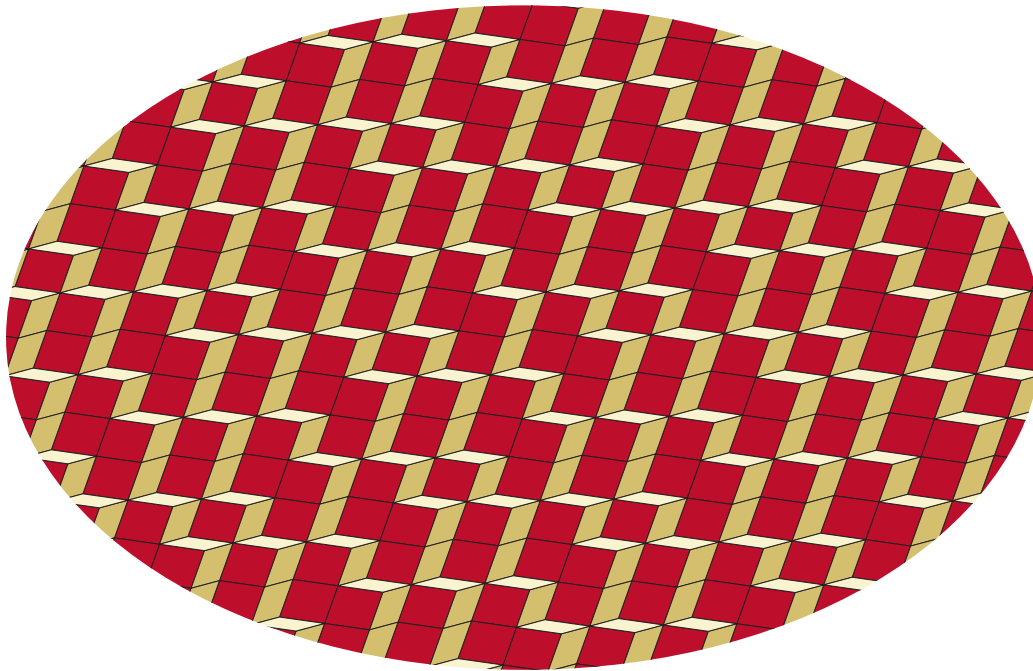
However, in 1966 Wang's student Berger showed there was a problem with this: he discovered a set of tiles that did tile the plane, but in a way that never repeated. In fact, the tiles could never be put together to form a patch that could be repeated: the tiles have to be continually put together in different configurations. Any algorithm of the sort outlined above would never finish. Berger had found the first example of an aperiodic tiling in 1966. He needed 20,426 different tiles to make it. Once the first example had been found, others quickly found aperiodic tilings using smaller sets of tiles. By 1971 Raphael Robinson had a tiling using just 6 tiles, but a few years later Roger Penrose had his, using just 2 tiles.

Less than 10 years later the patterns these tiles made had been found in nature as the positions of atoms in metal alloys. The 3 dimensional analogues of these 2-d tilings are now understood as key shapes in the structure of viruses and are being used to understand and predict virus evolution.

The quest for a monotile. Penrose's tiling is made of just two shapes of tile. At the moment, no one knows if there is a single shape of tile that tiles the plane but only aperiodically. Can you find one, or prove that there can be no such tile?

# Non-periodic slices

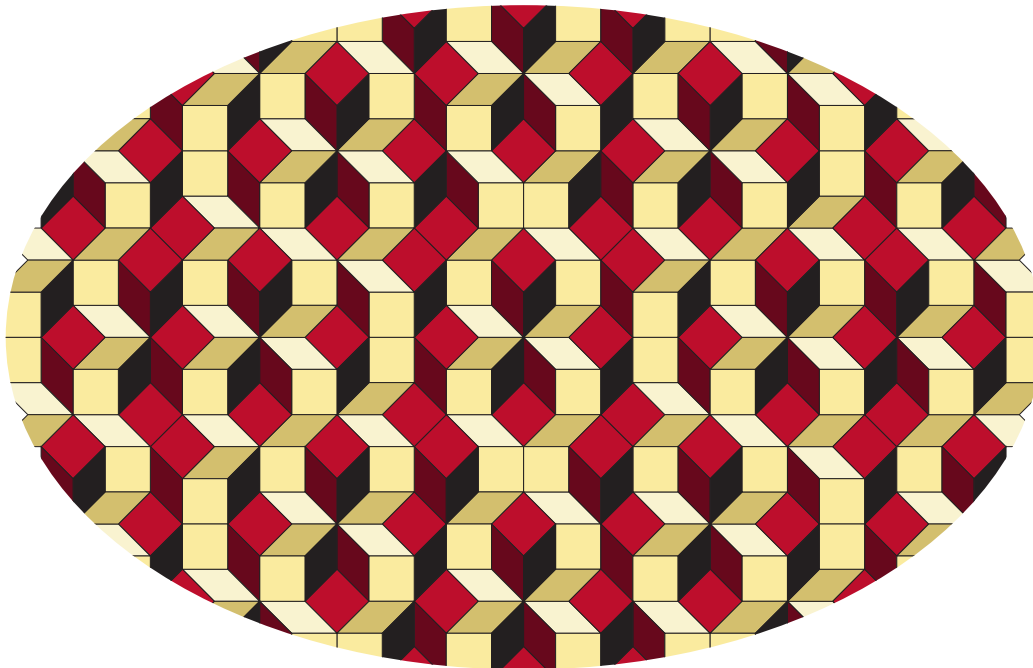
How can we easily generate non-periodic tilings? Here is a method that has proved to be very useful. Take a look at this:



Is this a picture of a stack of blocks, or is it a tiling made from 3 different shapes of tile?

Well it's both! One way to create unlimited examples of aperiodic tilings is to take a periodic tiling of a higher dimensional space - the picture above used a tiling of 3d space by cubes - and then taking a slice through it. So long as the slice is at an irrational slope with respect to the sides of the cubes the result is a non-periodic pattern of tiles.

Here is another example:



this time a 2d slice through a tessellation of 4d by 4 dimensional cubes. The Penrose tiling can be made as a slice through 5d space.



# New Maths, New Science!

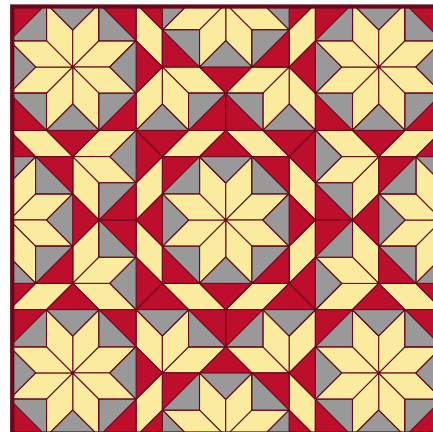
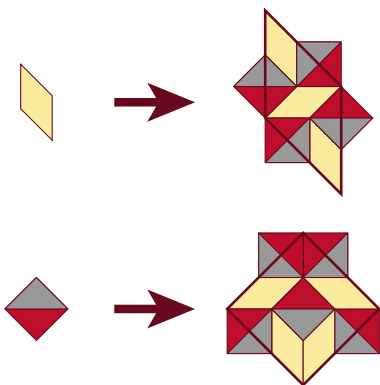
## Factsheet 7

### Tilings Zoo

We now have a large zoo of aperiodic tilings, though there are still very few simple ones. New examples with less than even ten tiles that are not related to old examples are still of great interest to mathematicians. A powerful method of finding aperiodic tilings was given by Goodman-Strauss in 1997. He proved that the tiles for any substitution tiling (see Factsheet 5) could be turned into an aperiodic set by adding jigsaw like notches to the edges and corners. Unfortunately in most cases this also vastly increased the number of tiles required. Aperiodicity is therefore still a very mysterious phenomenon. Here are some of the many substitution tilings that can be made into aperiodic tilings. For more examples see the *Tilings Encyclopedia*: <http://tilings.math.uni-bielefeld.de/>

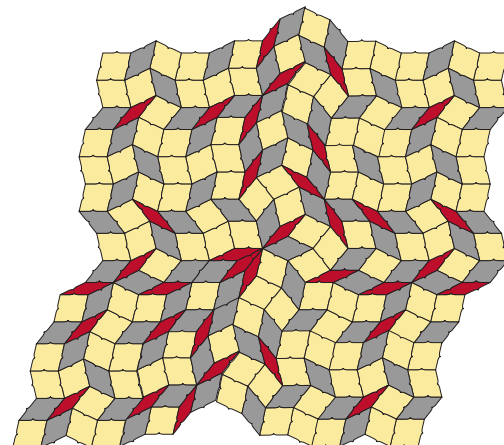
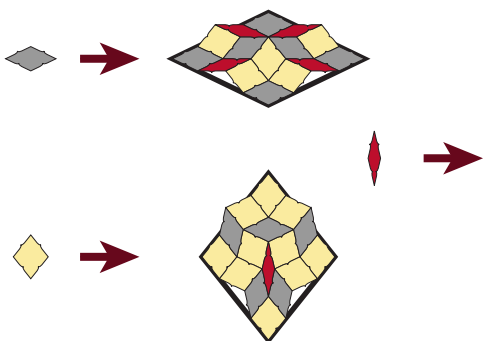
### Ammann-Beenker Tiling

This tiling can be constructed both as a substitution tiling, and as a slice through 4 dimensional space filled with 4d cubes. It is therefore related to the Penrose tiling, which can be made as a substitution tiling and as a slice of a periodic tiling of 5d space. It was first constructed in 1977 by amateur mathematician Robert Ammann, a US postal worker.



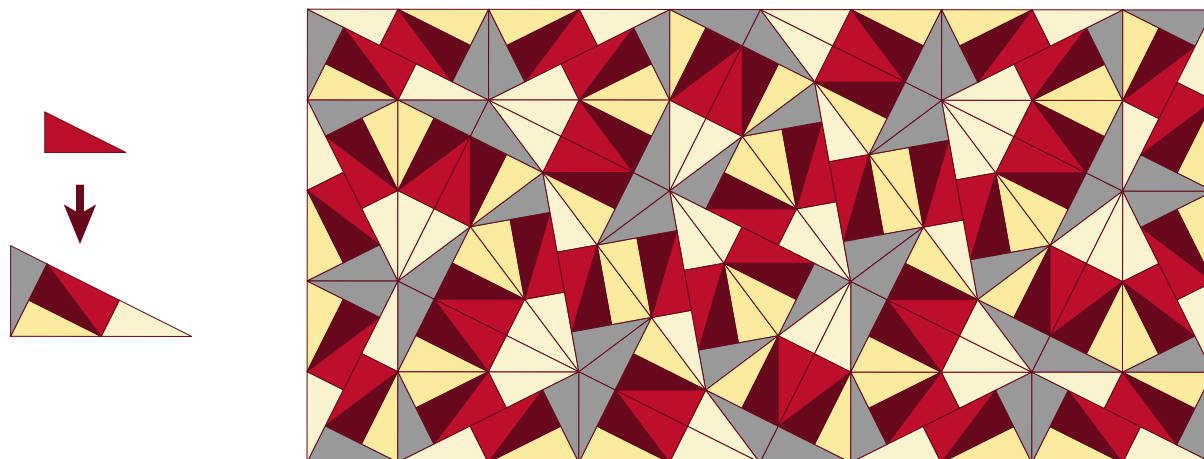
### Goodman-Strauss 7-fold rhomb Tiling

This beautiful tiling was discovered by Chaim Goodman-Strauss. It is part of a family of rhomb substitution tilings that have n-fold rotational symmetry for any n, discovered by Edmund Harriss in 2004.



## Pinwheel Tiling

Pinwheel: The Pinwheel tiling comes from a simple substitution rule on a single right angled triangle with side lengths 1, 2 and  $\sqrt{5}$ . Although it has only one shape of tile, this tile appears in an infinite number of rotations in the whole tiling. This complexity makes it hard to analyse and it is only in research carried out in the last year that we have begun to fully understand its properties. Nevertheless, it has already been used in architecture; the Federation Square buildings in Melbourne, Australia are built and decorated with a giant version of this tiling.



## Rauzy Tiling

In the pinwheel tiling and our original example of a substitution tiling, the chair tiling on Fact-sheet 5, the expanded tile was exactly divided into the new tiles. In the other example this is not the case. We can always change into a version of the tiling where the expanded tiles are divided. In most cases this is at the cost of having fractal boundaries, not straight lines, as shown in this Example discovered by Gerald Rauzy.

